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VISCOUS SPLIT ALGORITHMS FOR THE TIME DEPENDENT IMCOMPRESSIBLE NAVIER STOKES EQUATIONS

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RESEARCH AND TECHNOLOGY DEPARTMENT

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For problems involving the processes of convection and diffusion, a viscous split algorithm is one in which each process is solved separately. The emphasis of this paper is the application of this procedure to obtain numerical solutions to the incompressible Navier-Stokes equations. A second order Godunov method is used to solve the convection step of the algorithm. The diffusion step (Stokes equation) is solved by the Galerkin Finite Difference Method which projects the solution onto a descreetly divergence free space using local mesh basis functions. Numerical results for both steady and unsteady flows are presented.

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FOREWORD

This report describes an algorithm for the solution of the incompressible Navier-Stokes equations in which the convective and diffusive terms are treated independently. This treatment allows for the use of a second order Godunov method to discretize the nonlinear convection terms. The resulting scheme is stable for high Reynold's number flows.

I would like to thank Prof. Ivo Babuska of the University of Maryland, Drs. Jay M. Solomon and Stephen F. Davis of this Center, and Dr. John B. Bell of Lawrence Livermore National Laboratory for their useful ideas and comments concerning this algorithm, and Dr. Richard Lau for his support of this project.

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CHAPTER 1

INTRODUCTION

The major difficulty in designing algorithms for problems which have both convective and diffusive terms arises from the fact that the algorithm should work well for both parabolic problems (when diffusion dominates) as well as for hyperbolic problems (when convection dominates). In the diffusion dominant case, standard finite element or finite difference methods which yield space centered discretizations, coupled with a stable time discretization, will generally yield accurate results. However, central differencing of the convection terms will lead to an unstable (in space) scheme, evidenced by spurious oscillations. If the oscillations are local, this problem can be alleviated with an appropriate mesh refinement.[1] This procedure will fail if either the oscillations are global,[2] or they appear in smooth regions of the flow,[3] which could mislead one into refining the grid in the wrong area. One remedy to this problem is to incorporate a priori information about the solution into the difference scheme, which results in schemes which are nearly upwind when convection dominates and are nearly centered when diffusion dominates.[4] Although this procedure works well for one dimensional steady-state problems, its generalization to multidimensional problems has not yet been successful due to the complicated behavior of the solutions,[5] and the excessive diffusion resulting when the operators in each direction are treated as in the one dimensional case.[6] An alternative approach is to split the processes of convection and diffusion, and use independent stable and accurate schemes for each part.

Various studies have been performed concerning "viscous splitting" in References 7 through 10. When both the diffusive and convective steps are solved exactly, Beale and Majda[7] proved that the error is $O(\nu\tau^2)$ in L_2, where $\tau = \Delta t$ is the time step, and ν is the viscosity (diffusion coefficient). Characteristic Galerkin methods, which may be considered as viscous split algorithms using the method of characteristics for the convection step, applied to convection diffusion problems were analyzed in References 8 and 9. However, both treatments were only first order accurate in the time step, due to the use of the backward Euler method for integrating the diffusion step. A second order viscous split method was analyzed in Reference 10 for the solution to steady state one dimensional convection diffusion problems, and some of the ideas developed in that paper are used here as well.

This report is concerned with applying viscous splitting to the incompressible Navier-Stokes equations, employing a higher order Godunov method for the convection step. These methods have been demonstrated to be extremely successful for solving a wide variety of applications requiring solutions of hyperbolic systems (see e.g., References 11 and 12 and the references cited therein). We remark that the algorithm of Bell et al.,[13] which does not independently split the diffusion and

convection terms, is similar to the split algorithm employed here in the sense that both employ a Godunov scheme for the discretization of the convective terms as well as a similar projection scheme for the treatment of the incompressibility constraint.

CHAPTER 2

VISCOUS SPLITTING

The time dependent incompressible Navier-Stokes equations in a bounded domain Ω are

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{F}, \tag{2-1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2-1b}$$

with initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{2-1c}$$

and boundary conditions

$$\mathbf{B}(\mathbf{u}) = \mathbf{b}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \partial\Omega, \tag{2-1d}$$

where $\mathbf{u} = (u,v)$ is the velocity, p is the pressure, ν is the viscosity, and $\mathbf{F}$ is a source function. Equation (2-1b) is the incompressibility constraint and (2-1d) specifies boundary conditions of either Dirichlet and/or Neumann type on various parts of the boundary $\partial\Omega$. When the problem is non-dimensionalized by a characteristic length and velocity the Reynolds number Re is defined by

$$\text{Re} \approx \frac{1}{\nu}.$$

The solution of (2-1) will be approximated by solving the sub-problems of diffusion and convection separately. That is, (2-1) will be split into Stokes equation (diffusion)

$$\mathbf{w}_t - \nu\Delta\mathbf{w} = -\nabla p + \mathbf{F},$$

$$\nabla\cdot\mathbf{w} = 0,$$

$$\mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega,$$

$$\mathbf{B}(\mathbf{w}) = \mathbf{b}, \quad \text{for } \mathbf{x} \in \partial\Omega, \tag{2-2}$$

and a convection step

$$\mathbf{z}_t + (\mathbf{z}\cdot\nabla)\mathbf{z} = 0,$$

$$\mathbf{z}(\mathbf{x},0) = \mathbf{z}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{2-3}$$

where the treatment of the boundary conditions for the convection step will be discussed later. Let $S(\tau)$ denote the solution operator to (2-2) after a timestep τ. That is, $\mathbf{w} = S(\tau)\mathbf{w}_0$ is the solution to (2-2) at time $t = \tau$. Also, let $C(\tau)$ be such that $\mathbf{z} = C(\tau)\mathbf{z}_0$ is a solution to (2-3) at time $t = \tau$. The viscous split time stepping procedure for the approximate solution to (2-1) is

$$\mathbf{u}_n = S(\tau/2)C(\tau)S(\tau/2)\mathbf{u}_{n-1} \tag{2-4}$$

where $\mathbf{u}_n$ is an approximation to $\mathbf{u}(\mathbf{x}, n\tau)$. This type of splitting is often referred to as Strang splitting.[14] In addition, consider a split algorithm in which the diffusion operator is replaced by its Crank-Nicolson approximation $\tilde{S}$ (trapezoid rule in time). Then $\mathbf{w} = \tilde{S}(\tau)\mathbf{w}_0$ solves the problem

$$-\nu\frac{\tau}{2}\Delta\mathbf{w} + \mathbf{w} = \nu\frac{\tau}{2}\Delta\mathbf{w}_0 + \mathbf{w}_0 - \tau\nabla p^* + \tau\mathbf{F}^* ,$$

$$\nabla\cdot\mathbf{w} = 0, \quad \text{for } \mathbf{x} \in \Omega,$$

$$\mathbf{B}(\mathbf{w}) = \mathbf{b}(\mathbf{x}), \quad \text{for } \mathbf{x} \in \partial\Omega, \tag{2-5}$$

where the superscript "*" indicates an approximation at time $\frac{\tau}{2}$. In this case the splitting algorithm becomes

$$\bar{\mathbf{u}}_n = \tilde{S}(\tau/2)C(\tau)\tilde{S}(\tau/2)\bar{\mathbf{u}}_{n-1} = \left[\tilde{S}(\tau/2)C(\tau)\tilde{S}(\tau/2)\right]^n \mathbf{u}_0. \tag{2-6}$$

CHAPTER 3

SPATIAL DISCRETIZATION

As mentioned in the introduction, the main advantage of viscous splitting is that different and independent discretizations of the diffusion and convection steps may be used. In particular, it allows for the use of schemes of the type discussed in References 11 and 12, which have proven to be very successful for the solution of hyperbolic conservation laws.

In this report, a second order Godunov method with an approximate Riemann solver is employed for the discretization of the convective step. For the two dimensional hyperbolic system

$$u_t + F_x + G_y = 0 .$$

This algorithm is outlined in the following steps:

1. Compute and limit the slopes in the x and y directions for each component of the system. This produces a piecewise linear (not necessarily continuous) profile for u in each direction. The slope limiting is performed so that this piecewise linear approximation does not introduce spurious extrema.

2. Compute predicted values at the half time step. The predicted values are computed by differencing F and G at cell centers using values for u at cell edges from the linear profiles generated from the previous step. The predicted values are made approximately divergence free by adding the gradient of the pressure computed from the previous Stokes solve. This modification is necessary to preserve the formal second order accuracy of the viscous split routine.

3. Compute numerical fluxes along each edge by approximately solving a Riemann problem in each direction based on values from the prediction step. These fluxes must be consistent with the physical flux, i.e.,

$$F_h(u,u) = F(u),$$

where $F_h(u,v)$ is the numerical flux with states u and v at the left and right, respectively.

4. Correct the solution using the computed numerical fluxes.

The particular implementation employed for the results in this report is described in detail by Davis.[11] The slope limiting performed with these schemes can be shown to have the same effect as adding artificial viscosity only in regions where the concavity of the solution is changing abruptly, such as near shocks or interior or boundary layers. The solution of the Riemann problem associated with the convection terms of (2-3) results in an upwind scheme depending on the sign of the

velocity component. Along Dirichlet boundaries only the normal component of the velocity is prescribed, which is the usual condition for hyperbolic problems. The Courant-Friedrichs-Levy stability condition restricts the timestep by $c\tau/h = \lambda \le 1$, where $c = \max_{ij}(\,|\,2u_{ij}\,| + |\,2v_{ij}\,|\,)$. This restriction can be improved by a factor of two (c reduced by one half) using a non-conservative scheme, and with further modifications can be based on the maximum component instead of the sum (see e.g., Reference 13).

The Stokes solver of the Galerkin Finite Difference Method (GFDM) [3,5,15,16] was used for the discretization of $\tilde{S}$. We outline this method for the case of Dirichlet data. Further details as well as details of the appropriate modifications for more general boundary conditions may be found in Reference 3.

This method begins with an approximation of divergence at cell centers given by

$$(\mathbf{D}_h \mathbf{u})_{i+1/2,j+1/2} = \frac{1}{2h} \Big[u_{i+1,j+1} - u_{i,j+1} + u_{i+1,j} - u_{i,j} + v_{i+1,j+1} - v_{i+1,j} + v_{i,j+1} - v_{i,j} \Big] \quad (3\text{-}1)$$

and an approximation to the gradient defined at cell vertices by

$$(\mathbf{G}_h \phi)_{ij} = \frac{1}{2h} \begin{bmatrix} \phi_{i+1/2,j+1/2} - \phi_{i-1/2,j+1/2} + \phi_{i+1/2,j-1/2} - \phi_{i-1/2,j-1/2} \\ \phi_{i+1/2,j+1/2} - \phi_{i+1/2,j-1/2} + \phi_{i-1/2,j+1/2} - \phi_{i-1/2,j-1/2} \end{bmatrix} \quad (3\text{-}2)$$

for a scalar function ϕ defined at cell centers. The linear operator $\mathbf{D}_h$ maps a space of mesh vectors $\mathbf{V}_h$ defined at the vertices to a space of mesh scalars W_h defined at cell centers, and $\mathbf{G}_h$ maps W_h into $\mathbf{V}_h$. In Reference 15 these spaces are equipped with standard l_2 inner products such that $\mathbf{D}_h = \mathbf{G}_h^*$ holds, where $\mathbf{G}_h^*$ denotes the adjoint of $\mathbf{G}_h$. We remark that these definitions for $\mathbf{D}_h$ and $\mathbf{G}_h$ are the same discretizations that arise from a finite element approximation with bilinear velocities and piecewise constant pressures. An important observation employed by the GFDM is that $D_h = \{\Psi^{k+1/2,l+1/2}: k=1,...,m; l=1,...,n\}$ form a basis for the kernal of $\mathbf{D}_h$ where

$$\Psi_{ij}^{k+1/2,j+1/2} = \begin{Bmatrix} ((-1)^{l-j},(-1)^{k-i+1}) & \text{for } i=k,k+1; j=l,l+1 \\ (0,0) & \text{for all other } i,j \end{Bmatrix}. \quad (3\text{-}3)$$

The GFDM approximation to (2-5) is then to find $\mathbf{w}_h = \sum \alpha_{ij} \Psi_{i+1/2,j+1/2}$ such that

$$\left[-\nu \frac{\tau}{2} \Delta_h \mathbf{w}_h + \mathbf{w}_h, \Psi \right] = \left[\nu \frac{\tau}{2} \Delta_h \mathbf{w}_{0,h} + \mathbf{w}_{0,h} + \tau \mathbf{F}^*, \Psi \right] \quad \text{for each } \Psi \in D_h, \quad (3\text{-}4)$$

where Δ_h denotes for example the standard 5 point discrete Laplacian, and $\left[\cdot, \cdot \right]$ denotes an inner product on $\mathbf{V}_h$. The values for α_{ij} along Dirichlet boundaries are determined by the prescribed data provided certain compatibility conditions are met (again see References 3 or 15). The pressure is eliminated from this computation due to the fact that $\left[-\mathbf{G}_h p^*, \Psi \right] = \left[p^*, \mathbf{D}_h \Psi \right] = 0$ for each $\Psi \in D_h$. However, $\mathbf{G}_h p^*$ can be computed, for use in the convection (predictor) step, using the discretization of (2-5), after $\mathbf{w}_h$ is determined from (3-4).

CHAPTER 4

NUMERICAL RESULTS

Since the argument for order of accuracy was only formal, we first conduct a numerical convergence study on a smooth time dependent problem. The domain for this problem is the unit square with homogeneous Dirichlet data and initial conditions

$$u(x,y,0) = 0.5\sin(\pi y)\cos(\pi y)\sin^2(\pi x)$$

$$v(x,y,0) = -0.5\sin(\pi x)\cos(\pi x)\sin^2(\pi y) .$$

Solutions were computed on a sequence of uniform grids with N by N cells with N = 4, 8, ..., 64. The solutions were computed to time T = 0.125, with the timestep $\tau = h/2$ which corresponds to the Courant number $\lambda = .323$. Listed in Table 4-1 are the l_2 differences $E_N = ||u_N - u_{2N}||_{l_2}$, where the l_2 norm is computed on the coarser N by N grid. From this table the rates of convergence for Re=0 and Re=1000 are observed to be at least two. For Re= ∞, the rate is slightly less although the asymptotic rate has not been attained for this case.

TABLE 4-1. l_2 DIFFERENCES AND CONVERGENCE RATES

	Re=0 (Stokes)		Re=1000		Re=∞ (Euler)	
N-2N	E_N	Rate	E_N	Rate	E_N	Rate
4-8	.848e-2		.755e-2		.798e-2	
		2.22		1.46		1.45
8-16	.183e-2		.275e-2		.291e-2	
		2.09		2.36		2.26
16-32	.429e-3		.534e-3		.606e-3	
		2.03		2.02		1.59
32-64	.105e-3		.132e-3		.201e-3	

The algorithm is further demonstrated on a problem of 2-D channel flow over a full step at Re=100 (based on the inlet height and average inlet velocity). For this problem, the inlet velocity is parabolic, the channel has height 2 and length 30, the step is 1 unit high and 2 units wide, and the front face of the step is 6 units from the inlet. Figure 4-1 displays the streamline contours of the steady state solutions obtained using the viscous split algorithm and the unsplit GFDM[16] using

conservative centered differencing for the convective terms, on identical 30 by 16 grids. (The plots are stretched vertically by a factor of two for clarity.) In this case the viscous split solution produces results similar to streamline upwind type schemes, as described in References 15 through 17, in the sense that spurious oscillations near the front face of the step are eliminated and the length of the recirculation region behind the step is not decreased by excessive artificial diffusion.
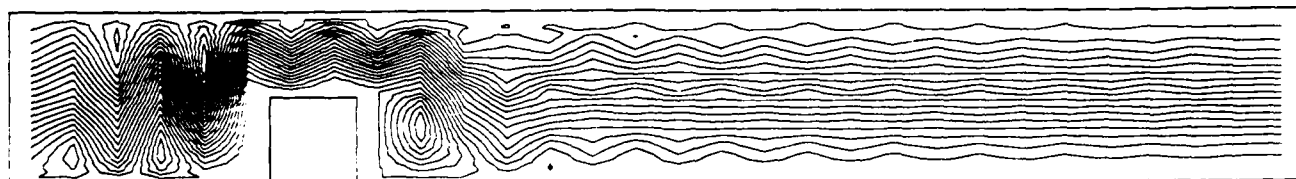
GFDM WITH CENTERED DIFFERENCE SCHEME
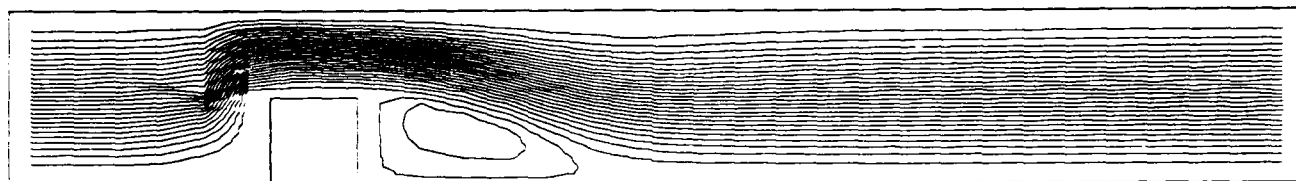
VISCOUS SPLIT GODUNOV SCHEME

FIGURE 4-1. COMPUTED STREAMFUNCTION CONTOURS OF STEADY STATE
SOLUTIONS AT Re=100 ON A 30 BY 16 GRID

Further tests on steady state problems for values of Re < 1000 were performed for flow over a rearward facing step and compared to the experimental results of of Reference 18 (as was done for the unsplit GFDM in Reference 16) with great success, but these results are not shown here. Instead, the time evolution of a flowfield at Re=10000 with initial velocities determined from the steady state Stokes solution is displayed in Figure 4-2. Although the 120 by 32 grid used in the calculation is insufficient to resolve all details of the flow, particularly near the step, we believe a reasonable depiction of the gross properties of the flow are preserved. The multiple recirculation regions have also been observed experimentally,[18] as well as for steady state computations at lower Reynolds numbers.[16]

The generation of stable (but not overdiffused) approximations on grids where all features of the flow are not resolved has important implications for developing an adaptive mesh refinement strategy, as demonstrated in References 2 and 10 for one dimensional steady problems. This feature, in addition to the property that an implicit time integration of the parabolic terms results in

a positive-definite symmetric system, makes viscous splitting an attractive approach for solving Navier-Stokes as well as other classes of problems with convective and diffusive terms.
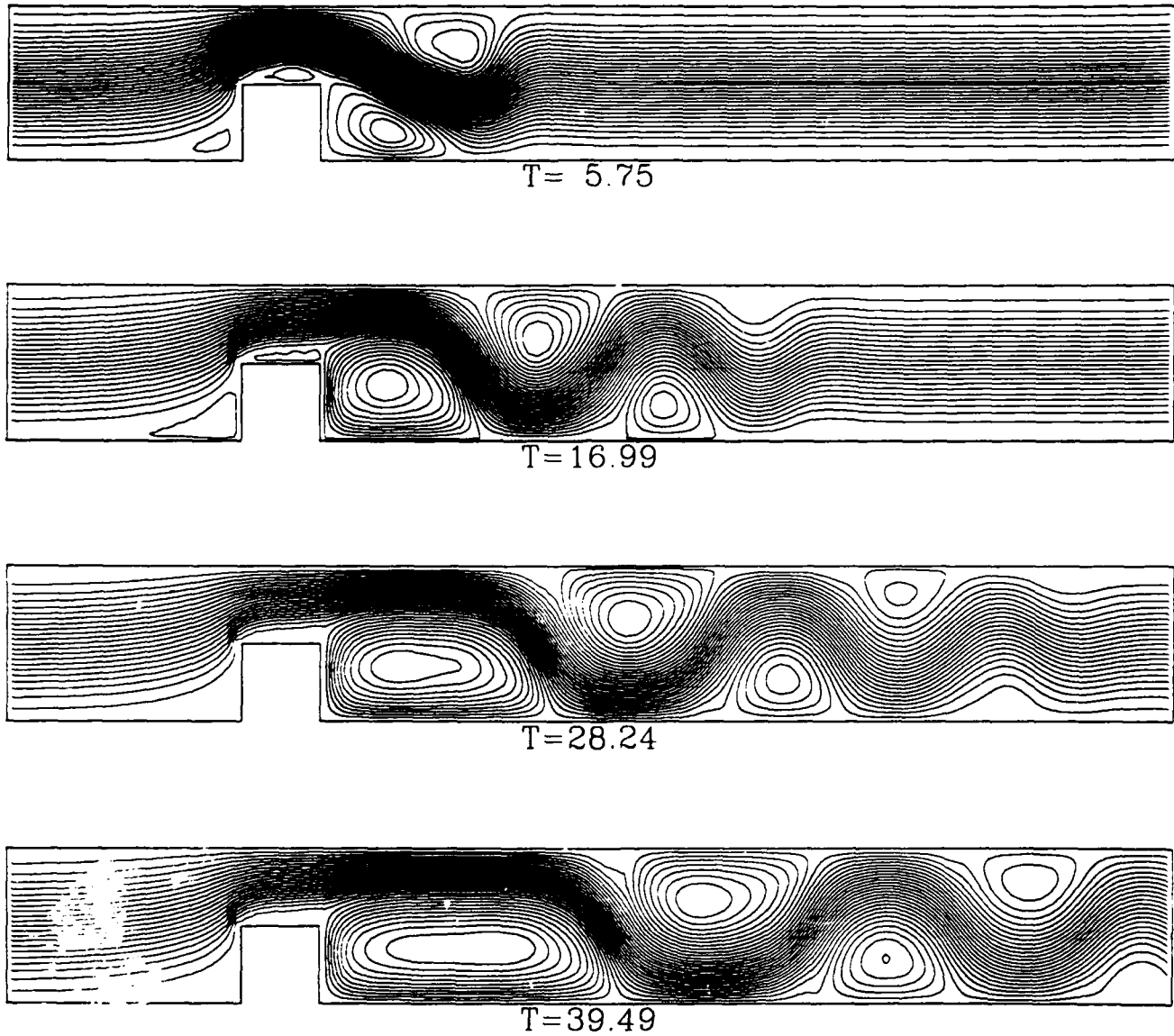

FIGURE 4-2. TIME EVOLUTION OF COMPUTED STREAMFUNCTION CONTOURS AT Re=10000 USING VISCOUS SPLITTING ON A 120 BY 32 GRID

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